Lecture 4 13-09-18

Integrating factor for non-exact equation:

Q: What can we do if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial t}$?

Idea: look for μ s.t.

$$\frac{\partial (\mu M)}{\partial y} = \frac{\partial (\mu N)}{\partial t}$$

There are two simple cases:

1) $\mu = \mu(t)$ only depending on t.

then we have

$$\mu \frac{\partial M}{\partial y} = \mu' N + \mu \frac{\partial N}{\partial t}$$

if $N \neq 0$ $\Rightarrow$ $\mu' = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N} \right) \mu$.

only depends on t   only depends on t

$\Rightarrow$ $K(t) = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N} \right)$ depends only on t.

$\Rightarrow$ one can solve $\mu = e^{\int K(t)\,dt}$

Case 2: $\mu = \mu(y)$ depends only on y

then similarily we have

$$\frac{d\mu}{dy} = \underbrace{\left(\frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y}}{M}\right)}_{\text{if only depends on } y\text{, then O.K.}} \mu \qquad \text{if } M \neq 0.$$

Idea: If M, N is NOT exact, try to see if

1) $N \neq 0$, and $K(t) = \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N}$ is independent of y

or 2) $M \neq 0$, and $H(y) = \dfrac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y}}{M}$ is independent of t

Then we can solve the integrating factor μ as

1) $\mu'(t) = K(t)\mu(t)$

2) $\mu'(y) = H(y)\mu(y)$

Example:

$$\underbrace{(3ty+y^2)}_{M(t,y)} + \underbrace{(t^2+ty)}_{N(t,y)}\frac{dy}{dt} = 0$$

$$\frac{\partial M}{\partial y} = 3t+2y \quad , \frac{\partial N}{\partial t} = 2t+y.$$

We consider

$$\left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N}\right) = \frac{t+y}{t(t+y)} = \frac{1}{t}$$

Independent of y

$$\left(\frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial y}}{M}\right) = \frac{-(t+y)}{y(3t+y)}$$

Hence we can set $K(t) = \frac{1}{t}$

$$\mu'(t) = K(t)\,\mu(t).$$

$\Rightarrow \mu(t) = te^c$ for some constant $c \in \mathbb{R}$.

Multiple the original equation by $\mu(t)$:

$$t(3ty+y^2) + t(t^2+ty)\frac{dy}{dt} = 0$$

We can solve for $\psi(t,y) = t^3 y + \frac{1}{2} t^2 y^2$

Hence the implicit solution to the ODE is

$$t^3 y(t) + \frac{1}{2} t^2 y(t)^2 = C$$

Rk: If neither condition hold, then we cannot force $\mu = \mu(t)$ or $\mu = \mu(y)$.

We force to take: $\mu = \mu(t,y)$, and look for

$$\frac{\partial(\mu M)}{\partial t} = \frac{\partial(\mu N)}{\partial y}$$

$$\frac{\partial \mu}{\partial t} M + \mu \frac{\partial M}{\partial t} = \frac{\partial \mu}{\partial y} N + \mu \frac{\partial N}{\partial y}$$

It is a partial different equation in μ.

which is a lot harder, therefore it is NOT a good idea in general to solve for factor μ.

§ Transformation method:

1. Bernolli equation:

$$\frac{dy}{dt} = p(t)\,y(t) + g(t)\,y^n. \quad \text{for } n \neq 0, 1.$$

Rewrite as: $\quad y^{-n}\frac{dy}{dt} - p(t)\,y^{1-n} = g(t).$

Idea: $\quad \frac{d\,y^{1-n}}{dt} = (1-n)\,y^{-n}\frac{dy}{dt}.$

Let : $\quad v(t) = y^{1-n}(t).$

then we have $\quad \frac{1}{1-n}\frac{dv}{dt} - p(t)\,v = g(t)$

or $\quad \frac{dv}{dt} - \underbrace{(1-n)p(t)}_{P(t)}\cdot v = \underbrace{(1-n)g(t)}_{\text{"} Q(t)}.$

Then we can solve for $v(t)$ using previous method:

$$\Rightarrow \quad V(t) = \frac{1}{\mu(t)}\left[\int Q(t)\,\mu(t)\,dt + c\right]$$

$$y(t) = \left(\frac{1}{\mu(t)}\left[\int Q(t)\,\mu(t)\,dt + c\right]\right)^{\frac{1}{1-n}}$$

2. $\quad f(t,y) = g(y/t)$ and we consider

$$y' = g(y/t).$$

In this case, we consider a new variable

$$v(t) = y(t)/t \quad \text{and} \quad v' = \frac{y'}{t} - \frac{y}{t^2} = \frac{y'-v}{t}.$$

$\therefore$ the equation become

$$v' + \frac{v}{t} = \frac{1}{t} g(v)$$

$$\Rightarrow \quad \boxed{\frac{1}{g(v)-v} v' = \frac{1}{t}} \quad \text{separable equation.}$$

Ex: $\frac{dy}{dt} = \frac{y-9t}{t-y} = \frac{\frac{y}{t}-9}{1-\left(\frac{y}{t}\right)}$.

Let $g\left(\frac{y}{t}\right) = \frac{\frac{y}{t}-9}{1-\frac{y}{t}}$. $\quad g(v) = \frac{v-9}{1-v}$.

$$\Rightarrow \frac{1}{g(v)-v} = \frac{1-v}{v^2-9} = \frac{1-v}{(v+3)(v-3)} = -\frac{2}{3}\frac{1}{(v+3)} - \frac{1}{3}\frac{1}{(v-3)}$$

Hence we have

$$\underbrace{\frac{-1}{t}}_{M(t)} + \underbrace{\frac{1-v}{(v+3)(v-3)}}_{N(v)} \frac{dv}{dt} = 0.$$

$$\rightsquigarrow \quad m(t) = -\log|t|, \qquad n(v) = \frac{-2}{3}\log|v+3| - \frac{1}{3}\log|v-3|.$$

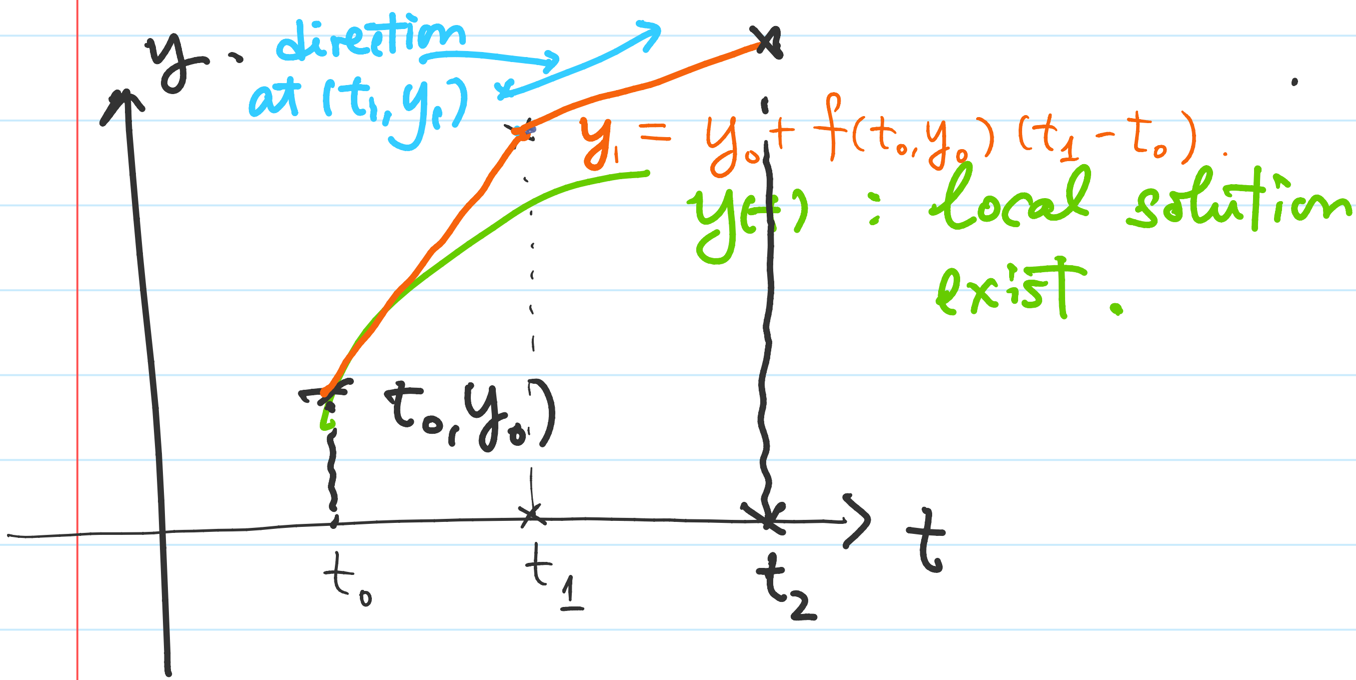
Plug into the general formula.

$$-\log|t| - \frac{2}{3}\log|v+3| - \frac{1}{3}\log|v-3| = C.$$

As a summary: we have deal with

Type	Method	form of the solution.
$y' = p(t)y + g(t)$	integrating factor	$y = e^{\int p(t)dt}\left[\int e^{-\int p(t)dt} g(t)\,dt + C\right]$
$M(t,y) + N(t,y)y' = 0$ $(M(t) + N(y)y' = 0)$	exact equation	$\psi(t, y(t)) = C.$ implicit solution.
$y' = p(t)y + g(t)y^n$	transformation $v = y^{1-n}$	$y = \left(\mu(t)^{-1}\int \mu(t) g(t)\,dt + c\right)^{\frac{1}{1-n}}$
$y' = g(y/t)$	$v = y/t$	$\frac{1}{g(v) - v}\frac{dv}{dt} = \frac{1}{t}$ separable equation $\longrightarrow$ solve

§ Euler's method : $y' = f(t,y)$.



Calculus Idea: the line $y_0 + f(t_0, y_0)(t - t_0)$ is a good approximation to $y(t)$ near (t_0, y_0).

Step by step:

1) fixed $t_1 > t_0$

find $y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$

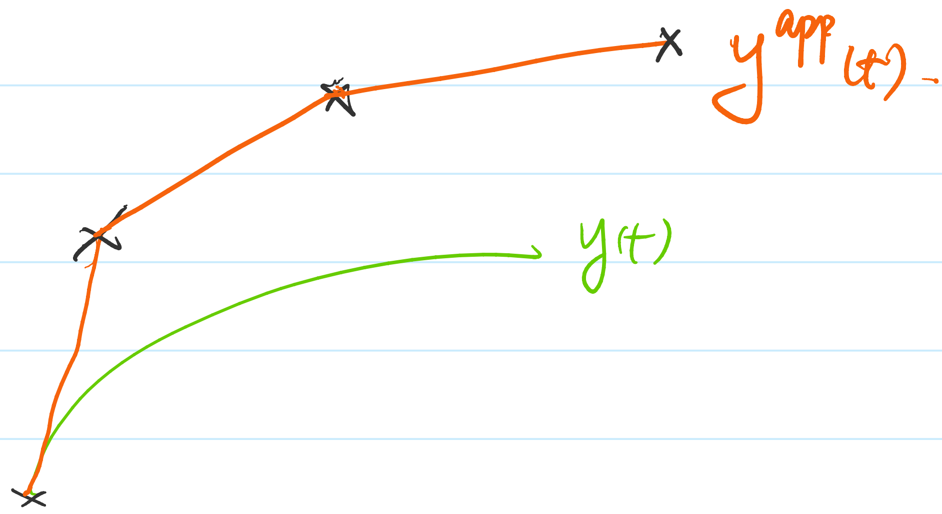
2) fixed $t_2 > t_1$ and

find $y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$

⋮

n) fixed $t_n > t_{n-1}$

find $y_n = y_{n-1} + f(t_{n-1}, y_{n-1})(t_n - t_{n-1})$



We obtain an piecewise Linear curve $y^{app}(t)$ to approximate y.

<u>Drawback</u>: NOT very accurate as t increase!